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The physical meaning of the embedded effect in the quantum submanifold system

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Abstract. In quantum mechanics on a submanifold, it is known that when the submanifold has an extrinsic curvature, an effective potential appears in the Schrödinger equation even if it does not curve intrinsically. Recently Ikegami *et al* applied the Dirac quantization scheme for a constrained system to submanifold physics and found that there is an anomalous correspondence between the quantum and the classical mechanics. In this paper, we show the physical meaning of the origin of it through the polar representation and then the results of Ikegami *et al* are naturally understood.

1. Introduction

In elementary particle physics and quantum gravity, there are many studies of quantum physics on a manifold (Birrell and Davies 1982), in which the intrinsic curvature plays the most important role. Consideration of the extrinsic curvature is sheer nonsense because the intrinsic property of the manifold does not depend upon whether it is embedded or not. Even after quantization, it is assumed that this remains true because the outer space of the universe should not have an effect on the inner space.

In this decade, the quantum system on a submanifold was studied in condensed matter physics. In the quantum submanifold system, the extrinsic curvature is more important than the intrinsic one. If a submanifold in $\mathbb{R}^3$ has an extrinsic curvature, the curvature sometimes makes an attractive potential appear in the Schrödinger equation. This effective potential appears even though there is no intrinsic curvature; a torus or a space curve. Hereafter we call it the embedded potential. It comes from a geometrical correction at the quantum level and has the form

$$V_{\rm eff}^{\rm 2D} = -\frac{\hbar^2}{2m}\left((\tfrac{1}{2}{\rm tr}_2(\Gamma^\alpha{}_{3\beta}))^2 - {\rm det}_2(\Gamma^\alpha{}_{3\beta})\right) \qquad (1.1)$$

where $-\Gamma^\alpha{}_{3\beta}$ is the Weingarten map. It was derived by da Costa (1981) using the operator formalism, and by Matsutani (1992a, 1993) using the path integral method. In both methods, a confinement potential was introduced. Since, in Euclidean space, the quantum physics is well defined, one can use conventional quantum mechanics. Taking the squeezed limit of the potential, which confines a particle on the submanifold, the embedded potential was obtained. These embedded potentials from both methods agree.

This effect was applied to a particle on a rod and it was then found that this effect is closely related to soliton physics (Matsutani and Tsuru 1991, 1992, Matsutani 1992b). Furthermore there are many other applications; a curved quantum wire (Duclos and Exner 1991) and a curved wave guide (Miyagi 1989).

Recently Ikegami *et al* (1992) studied the quantum submanifold system using the Dirac quantization scheme for a constrained system (Dirac 1964). After classically constraining a particle on the submanifold, they then quantized the system. Then they pointed out that there is an anomalous correspondence between the classical and the quantum mechanics in the Dirac scheme for the submanifold quantum system. Let q^3 be a function of $\mathbb{R}^3$ and $q^3 = 0$ express a surface in $\mathbb{R}^3$. They dealt with it as two different constrained systems. First, they made the particle satisfy $q^3 = 0$ and quantized it (they called this first situation D-case). Second, they considered it under the condition $\dot{q}^3 = 0$ (they called this second case $\dot{D}$-case). Due to the requirement that the condition should preserve a consistency for the time development, the D-case contains the constraint condition $\dot{q}^3 = 0$. Hence the conditions in the D-case have a symmetry in the phase space and look natural. Thus in elementary particle physics, the D-case is well established while the $\dot{D}$ is not so well; e.g. the D-case is studied by Marinov and Terentyev (1979), and Fukutaka and Kashiwa (1987) for a sphere in terms of the path integral method and by Ogawa *et al* (1990) for a general submanifold in terms of the original Dirac scheme. Both the D and $\dot{D}$-cases generate the same result in the classical region. However after one quantizes them, they have different embedded potentials; for the D-case

$$V_D^{2\mathrm{D}} = \frac{\hbar^2}{8m}\left(1 + (\xi_1)^2\right)\left(\mathrm{tr}_2({\Gamma^\alpha}_{3\beta})\right)^2 \tag{1.2}$$

and for the $\dot{D}$-case

$$V_{\dot{D}}^{2\mathrm{D}} = -\frac{\hbar^2}{8m}\left(\xi_2 \mathrm{tr}_2(({\Gamma^\alpha}_{3\beta})^2) - \xi_3(\mathrm{tr}_2({\Gamma^\alpha}_{3\beta}))^2\right) \tag{1.3}$$

where the ξ's are real parameters, which come from the ambiguity in the ordering problem in the quantization. It is remarked that the embedded potential in the D-case (1.2) disagrees with that of the operator formalism (1.1) for any ξ_1. Ogawa (1992) first pointed out this difference between the D-case (1.2) and the operator formalism (1.1). On the other hand, the embedded potential in the $\dot{D}$-case (1.3) is in agreement with the conventional one for a physical choice; $\xi_2 = 2$ and $\xi_3 = 1$. In other words, the quantization depends upon the choice of the constraint conditions and the D-case does not reflect the real physics. It looks anomalous, and Ikegami *et al* indicated the physical meaning of this anomalous result. However it is not so clear why the $\dot{D}$-case is more natural than the D-case.

In this paper, by means of the polar representation (Bohm 1952, Dirac 1958, Sakurai 1985), we will clarify this phenomenon and reveal the reason why the $\dot{D}$-case survives under the quantization and expresses physical states. Furthermore, we try to show a more intuitive physical meaning of the embedded potential and the relations between the path integral method, the operator formalism and the Dirac scheme on the submanifold system.

2. Polar-representation of the Schrödinger equation

First of all, we consider the Schrödinger equation on the flat space $\mathbb{R}^3$ and express it in terms of the Cartesian coordinate (t, x^i), $i = 1, 2, 3$

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\delta^{ij}\partial_i\partial_j\psi + V\psi \tag{2.1}$$

where $\partial_t := \partial/\partial t$ and $\partial_i := \partial/\partial x^i$. Let us use the polar (Madelung) representation

$$\psi(x,t) = R(x,t)\exp(\mathrm{i}S(x,t)/\hbar) \tag{2.2}$$

where R and S are real valued functions. Then the Schrödinger equation (2.1) becomes

$$\partial_t R = -\frac{1}{2m}[R\delta^{ij}\partial_i\partial_j S + 2\delta^{ij}\partial_i R\partial_j S] \tag{2.3}$$

$$\partial_t S + \frac{1}{2m}\delta^{ij}(\partial_i S)(\partial_j S) + V - \frac{\hbar^2}{2m}\frac{\delta^{ij}\partial_i\partial_j R}{R} = 0. \tag{2.4}$$

The first equation (2.3) indicates the continuity equation when we define $j_i := \rho\partial_j S/m$ and $\rho := R^2$

$$\partial_t\rho + \partial_i\left(\delta^{ij}\frac{1}{m}\rho\partial_j S\right) = 0. \tag{2.5}$$

Next we consider the second equation (2.4) (Bohm 1952, Dirac 1958, Sakurai 1985). It is known that the classical Hamilton–Jacobi equation (CHJE) may be written as (Arnold 1989)

$$\partial_t S_c + \frac{1}{2m}\delta^{ij}(\partial_i S_c)(\partial_j S_c) + V = 0 \tag{2.6}$$

for the classical Hamiltonian

$$H_c = \frac{1}{2m}\delta^{ij}p_i p_j + V. \tag{2.7}$$

According to the relation between (2.6) and (2.7), $\partial_i S_c$ corresponds to the classical momentum.

Comparing (2.4) and (2.6), the second equation (2.4) can be regarded as a kind of Hamilton–Jacobi equation with the quantum correction

$$V_Q = -\frac{\hbar^2}{2m}\frac{\delta^{ij}\partial_i\partial_j R}{R}. \tag{2.8}$$

Let us call the second equation (2.4) the quantum Hamilton–Jacobi equation (QHJE) and the extra term (2.8) the quantum potential (QP). In the QHJE, $\partial_i S$ seems to indicate the quantum momentum. When we take the classical limit $\hbar \to 0$, the QP appears to vanish and the QHJE seems to agree with the CHJE (Dirac 1958, Bohm 1952). There, thus, seems to exist a classical–quantum correspondence; $\partial_i S_c \Leftrightarrow \partial_i S$ and $V \Leftrightarrow V + V_Q$.

However, it is known that R sometimes contains $(x/\hbar)^2$ and the QP survives in the classical limit. In order to simplify the problem, we deal with a one-dimensional system for a while. For example, in the harmonic potential case, the QP does not vanish as $\hbar \to 0$ (Song Ling 1992). For the $V \equiv 0$ situation (free case), it is known that $\psi = \exp(\mathrm{i}xp/\hbar)$ and $\psi = \cos(\mathrm{i}xp/\hbar)$ satisfy the same equation (2.4). While the exponent solution has the correspondence through the fact that $\partial_x S$ is just the momentum eigenvalue p and the QP vanishes, the cosine solution indicates that $\partial_x S \equiv 0$ and the QP remains as the kinetic term in the classical limit. Hence the appearance of the dependence of $\hbar$ does not reflect the real physical situation. This is natural because the QP contains a second-order derivative, i.e. the square of the momentum operator. In order to clarify the problem, let us consider a simpler example; a symmetrical infinite box potential, where $V(x) = 0$ for $x \in (-d, d)$ and $V(x) = V_0$ with $V_0 = \infty$ for $x \notin (-d, d)$. Its solution is $\psi = \cos(\mathrm{i}xp_n/\hbar)$ with the quantized momentum p_n. Then as mentioned above, the quantum and classical correspondence through the QHJE breaks down. In other words, the QP does not vanish for $\hbar \to 0$ and $\partial_i S \equiv 0$. Even for $V_0 < \infty$, it is true. An intuitive reason why it breaks down is that the classical theory is a local theory while the quantum theory is a global theory. The penetration of the wave function to the outer space of the box ($x \notin (-d, d)$) is a quantum effect. (The phase S is suppressed and only R survives there. R indicates a quantum effect.) The boundary condition has an effect on the wave function over all the region. In other words, the boundary condition breaks the translational invariance and then the generator of the invariance (the momentum operator) behaves peculiarly over all the region; the correspondence between $\partial_x S$ and the momentum p breaks down. This phenomenon is also found in a periodic boundary problem; e.g. on a topological connected circle S^1 with circumference $2\pi r$, the translational invariance is also modified and the eigenvalue of the momentum operator is discretized. Then $\partial_x S_c \in \mathbb{R}$ while $\partial_x S$ is expressed by an integer.

Consequently since the difference between the local and the global theories is too crucial for a space with a boundary, especially for the bound states, we cannot continue to deal intuitively with the correspondence between the CHJE and QHJE there.

However we can avoid these problems. For example, we could redefine S and extend it to the complex valued function with a quantized condition like the WKB method (Bohm 1951, Vigier 1989). Another possibility is that we could consider (2.3) and (2.4) only over an open space without bound states.

In this paper, for the sake of simplicity, we will employ the latter method. Then we can avoid the boundary value problem. Furthermore, we restrict ourselves to considering only the subset of its solutions whose $\partial_x S$ can be regarded as a momentum; for example in the $V \equiv 0$ case, we deal with $\psi = \exp(\mathrm{i}xp/\hbar)$ rather than $\psi = \cos(\mathrm{i}xp/\hbar)$. We note that this restriction is not so rigid since in terms of the linearity of the equation, we can obtain the cosine solution by superposing the exponent solutions. Then we can go along with the intuitive correspondence between the classical and the quantum mechanics. In other words, the origin of the $\hbar$ fluctuation can be regarded as the QP and we can consider $\partial_x S$ as the momentum of the system. Consequently under the restrictions, the QHJE can be regarded as the deformation of the CHJE.

We also note that since we deal with an open base space, we are now looking at a scattering problem.

Next we will show the relation between the path integral and the QHJE (Dirac 1958, Schulman 1981). It is known that a solution of the CHJE is the action integral of the system (Arnold 1989)

$$S_0[x] = \int \mathrm{d}t\, \tfrac{1}{2} m \delta_{ij} \dot{x}^i \dot{x}^j - V. \tag{2.9}$$

On the other hand, using the path integral representation the wave function ψ is expressed by (Feynman and Hibbs 1965)

$$\psi(x, t) = \int \mathrm{d}x_\mathrm{i}(t_\mathrm{i})\, Z(x, t; x_\mathrm{i}, t_\mathrm{i}) \psi(x_\mathrm{i}, t_\mathrm{i}) \tag{2.10}$$

where the 'i' index indicates an initial state and

$$Z(x, t; x_\mathrm{i}, t_\mathrm{i}) = \int_{x_\mathrm{i}(t_\mathrm{i})}^{x(t)} Dx\, \exp(\mathrm{i}S_0[x]/\hbar). \tag{2.11}$$

Let us define the effective action $S_{\rm eff}$ by

$$S_{\rm eff}(x, t; x_{\rm i}, t_{\rm i}) := -{\rm i}\hbar \log Z. \tag{2.12}$$

It is expressed by

$$S_{\rm eff} = \langle S_0 \rangle_{\rm PI} - {\rm i}\hbar S_{\rm ent} \tag{2.13}$$

where $\langle\rangle_{\rm PI}$ means

$$\langle S_0 \rangle_{\rm PI} := \int Dx(t)\, S_0 \exp({\rm i}S_0/\hbar)/Z(x, t; x_{\rm i}, t_{\rm i}). \tag{2.14}$$

Then ψ becomes

$$\psi = \int {\rm d}x_{\rm i} \exp({\rm i}\langle S_0 \rangle_{\rm PI}/\hbar + S_{\rm ent})\psi(x_{\rm i}, t_{\rm i}). \tag{2.15}$$

It is known that in the path integral representation, the quantum and statistical mechanics have a correspondence if we interpret ${\rm i}\hbar$ as the absolute temperature and *vice versa* (Feynman and Hibbs 1965). According to this analogy, it turns out that $S_{\rm ent}$ seems to play a similar role to the entropy. In other words in the path integral representation of statistical physics, the entropy implies the volume of the allowed regions in the same way as $S_{\rm ent}$; in the classical limit $\hbar \to 0$, corresponding to the low temperature limit, the physically allowed region is that where the exponent $S_0/\hbar$ is minimum. Then the path is fixed and the volume of it, or $S_{\rm ent}$, vanishes. On the other hand, in the other limit $\hbar \to \infty$ which corresponds to the high temperature limit, $S_0/\hbar \to 0$ and the volume of the region or $S_{\rm ent}$ becomes sufficiently large. Accordingly $S_{\rm ent}$ indicates the quantum fluctuation and prefers the random state.

If we define $S_R := \log R$

$$\psi = {\rm e}^{{\rm i}S/\hbar + S_R}. \tag{2.16}$$

Roughly speaking, $S_{\rm ent}$ and S_R behave similarly in the quantum mechanics, if we approximate S by $\langle S_0 \rangle_{\rm PI}$. Actually, the QP (2.8), which indicates the quantum effect in the QHJE, consists of only S_R.

3. Submanifold quantum mechanics

In this section, we will confine a particle onto the two-dimensional (2D) surface Σ embedded in $\mathbb{R}^3$ (da Costa 1981, Matsutani 1992a, 1993). In order to keep the intuitive correspondence between the QHJE and the CHJE, we assume that Σ is open and homeomorphic to $\mathbb{R}^2$ and towards infinity, it approaches flat. First of all, we will define the geometry of the system we consider. Let the middle part of the Greek alphabet used as indices $(q^\mu, q^\nu, \ldots)$ indicate the curved system; $\mu = 1, 2, 3$. The relation between the Cartesian and the general coordinates is given through the dreibein

$$e^i{}_\mu := \partial_\mu x^i \tag{3.1}$$

where $\partial_\mu := \partial/\partial q^\mu$. The metric is written as

$$g_{\mu\nu} := \delta_{ij} e^i{}_\mu e^j{}_\nu. \tag{3.2}$$

Let the first and the second coordinates indicate the position attached on Σ. The normal unit vector of Σ is denoted by e_3. The confinement potential V is given along Σ and constrains the particle to be on Σ. Let us assume that V has the form, $V^{2D}_{\rm conf}(q^3) := \frac{1}{2}m\omega^2(q^3)^2$ for large ω, where q^3 is the normal coordinate of Σ. As we mentioned in the introduction, $q^3 = 0$ indicates the surface Σ. Hence we consider only the vicinity of Σ.

Because we wish the 3D metric $g_{\mu\nu}$ (3.2) around Σ to be expressed by the variables of Σ, we will consider the geometry in the vicinity of Σ. Let a position on Σ be denoted by $\boldsymbol{r}(q^1, q^2)$. We can express a point $\boldsymbol{x} := (x^1, x^2, x^3)$ around Σ in terms of the curved system

$$\boldsymbol{x}(q^\mu) = \boldsymbol{r}(q^\alpha) + q^3 \boldsymbol{e}_3. \tag{3.3}$$

The start of the Greek alphabet used as indices $(q^\alpha, q^\beta, \ldots)$ span from one to two. We define the zweibein along Σ as $b^i{}_\alpha := \partial r^i/\partial q^\alpha$ and the covariant derivative D_α as $D_\alpha \boldsymbol{X} := \partial_\alpha \boldsymbol{X} - \langle \partial_\alpha \boldsymbol{X}, \boldsymbol{e}_3\rangle \boldsymbol{e}_3$ for a vector $\boldsymbol{X}$. Here $\langle , \rangle$ denotes the canonical inner product. The 2D Christoffel symbol is thus defined as $D_\alpha \boldsymbol{b}_\beta = \Gamma^\gamma{}_{\beta\alpha} \boldsymbol{b}_\gamma$. The second fundamental form (Guggenheimer 1963) defined by $\Gamma^3{}_{\beta\alpha} := \langle \boldsymbol{e}_3, \partial_\alpha \boldsymbol{b}_\beta\rangle$ is expressed by

$$\Gamma^3{}_{\beta\alpha} = -\Gamma^\gamma{}_{3\alpha}\eta_{\gamma\beta} \tag{3.4}$$

where $\eta_{\alpha\beta} := \delta_{ij} b^i{}_\alpha b^j{}_\beta$ and $-\Gamma^\gamma{}_{\beta 3} := \langle \boldsymbol{b}_\gamma, \partial_3 \boldsymbol{b}_\beta\rangle$ is the Weingarten map. Therefore we can express $e^i{}_\mu (= \partial x^i/\partial q^\mu)$ around Σ in terms of $b^i{}_\alpha$

$$e^i{}_\alpha = b^i{}_\alpha + q^3 \Gamma^\beta{}_{3\alpha} b^i{}_\beta. \tag{3.5}$$

The 3D metric $g^{\mu\nu}$ around Σ can be found using (3.2) and $g := \det(g_{\mu\nu})$ becomes

$$g = \eta\zeta \qquad \zeta^{1/2} := \left(1 + \mathrm{tr}_2(\Gamma^\alpha{}_{3\beta})q^3 + \det{}_2(\Gamma^\alpha{}_{3\beta})(q^3)^2\right). \tag{3.6}$$

Here tr_2 and $\det_2$ are the 2D trace and determinant. These values are known as the mean and Gaussian curvatures on Σ (Guggenheimer 1963).

As we finish the geometrical preliminary, we will consider the quantum mechanics around Σ. In terms of the curved coordinate system, the Schrödinger equation becomes

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m} g^{-1/2}\partial_\mu g^{1/2} g^{\mu\nu}\partial_\nu\psi + V^{2D}_{\rm conf}\psi \tag{3.7}$$

with the confinement potential $V^{2D}_{\rm conf}(q^3) := \frac{1}{2}m\omega^2(q^3)^2$. Though along the normal direction the system is a bound state, along Σ it is an open space. Hence we can continue to deal with the polar representation formally. In terms of the polar representation, the Schrödinger equation becomes

$$\partial_t R = -\frac{1}{2m}[R g^{-1/2}\partial_\mu g^{\mu\nu} g^{1/2}\partial_\mu S + 2g^{\mu\nu}\partial_\mu R\partial_\nu S] \tag{3.8}$$

$$\partial_t S + \frac{1}{2m} g^{\mu\nu}(\partial_\mu S)(\partial_\nu S) + V^{2D}_{\rm conf} - \frac{\hbar^2}{2m}\frac{g^{-1/2}\partial_\mu g^{\mu\nu} g^{1/2}\partial_\nu R}{R} = 0. \tag{3.9}$$

The first equation also indicates the continuity equation if we define $j^\mu := g^{\mu\nu}\rho\partial_\nu S/m$ and $\rho = R^2$ (Landau and Lifshitz 1962)

$$\partial_t g^{1/2}\rho + \partial_\mu\left(\frac{1}{m}\rho g^{1/2} g^{\mu\nu}\partial_\nu S\right) = 0. \tag{3.10}$$

It is known that coordinate transformations in quantum mechanics needs some subtle treatment (Sakita 1985, Dirac 1958). Since the probability is expressed by $(\psi_1|\psi_2) := \int d^3x\, \psi_1^*(x) \cdot \psi_2(x)$ in the Cartesian coordinate, it becomes $(\psi_1|\psi_2) = \int d^3q\, g^{1/2}\psi_1^*(q) \cdot \psi_2(q)$ in the curved coordinate system. In general, the Jacobian impedes the Hermiticity of the natural differential operator. It is equivalent to the fact that ∂_i is the Killing vector in $\mathbb{R}^3$ while in general ∂_μ is not.

In our problem, we wish to separate the equation along the normal direction from the Schrödinger equation (3.7). Then the normal dynamics will be expressed by

$$\mathrm{i}\partial_t \psi_\mathrm{N} = \frac{1}{2m}\hat{p}_3^2\psi_\mathrm{N} + V_\mathrm{conf}^\mathrm{2D}\psi_\mathrm{N}. \tag{3.11}$$

However due to the Jacobian, $-\mathrm{i}\hbar\partial_3$ does not agree with the momentum operator $\hat{p}_3$. In other words, after confinement we expect that the probability density along Σ should be $(\phi_T^* \cdot \phi_T)(q^1, q^2) := \int d(q^3)\, (\phi^* \cdot \phi)(q^1, q^2, q^3)$. In order to get the probability density and for the derivative operator to agree with the momentum operator, we will deform the Hilbert space and we define a new wave function (da Costa 1981)

$$\phi := \zeta^{1/4}\psi \qquad \text{and} \qquad r = \zeta^{1/4}R. \tag{3.12}$$

Then the momentum operator $\hat{p}_3$ is identified with $-\mathrm{i}\hbar\partial_3$.

The continuity equation (3.10), thus becomes

$$\partial_t\tilde{\rho} + \partial_\mu\left(\frac{1}{m}\tilde{\rho}g^{\mu\nu}\partial_\nu S\right) = 0 \tag{3.13}$$

where $\tilde{\rho} := \eta^{1/2}r^2$. On the other hand, the QHJE (3.9) becomes

$$\partial_t S + \frac{1}{2m}g^{\mu\nu}(\partial_\mu S)(\partial_\nu S) + V_\mathrm{conf}^\mathrm{2D} - \frac{\hbar^2}{2m}\frac{\zeta^{1/4}g^{-1/2}\partial_\alpha g^{\alpha\beta}g^{1/2}\partial_\beta\zeta^{-1/4}r}{r}$$
$$- \frac{\hbar^2}{2m}\frac{\partial_3^2 r}{r} - \frac{\hbar^2}{2m}\left[\frac{3}{8}\frac{1}{\zeta^2}(\partial_3\zeta)^2 - \frac{1}{4}\frac{1}{\zeta}\partial_3^2\zeta\right] = 0. \tag{3.14}$$

Let us consider the effect on $V_\mathrm{conf}^\mathrm{2D}$. After $\omega \to \infty$, we can separate the equations to normal and horizontal parts (da Costa 1981). As we know that the solution of the harmonic potential for the lowest state, $S_\mathrm{N} = \hbar\omega t/2$ and $r_\mathrm{N} = (m\omega/\pi\hbar)^{1/4}\exp(-m\omega(q^3)^2/2\hbar)$ when we assume that S is written as $S = S_\mathrm{2D}(t, q^\alpha) + S_\mathrm{N}(t, q^3)$, and $r = r_\mathrm{2D}(t, q^\alpha)r_\mathrm{N}(q^3)$. The QHJE along the normal direction becomes

$$\partial_t S_\mathrm{N} + \frac{1}{2m}(\partial_3 S_\mathrm{N})(\partial_3 S_\mathrm{N}) + V_\mathrm{conf}^\mathrm{2D} - \frac{\hbar^2}{2m}\frac{\partial_3^2 r_\mathrm{N}}{r_\mathrm{N}} = 0. \tag{3.15}$$

Because $\partial_3 S_\mathrm{N} \equiv 0$ and r_N contains a $(q^3/\hbar)^2$ term, the intuitive correspondence between the CHJE and the QHJE is broken along the normal direction.

Next we consider the quantum equation along Σ. By integrating (3.13) over q^3, we obtain the 2D ordinary continuity equation

$$\partial_t\rho_\mathrm{2D} + \partial_\alpha\left(\frac{1}{m}\rho_\mathrm{2D}\eta^{\alpha\beta}\partial_\beta S_\mathrm{2D}\right) = 0 \tag{3.16}$$

where $\rho_{2D} := (r_{2D})^2$. The QHJE along the surface Σ becomes

$$\partial_t S_{2D} + \frac{1}{2m}\eta^{\alpha\beta}(\partial_\alpha S_{2D})(\partial_\beta S_{2D}) - \frac{\hbar^2}{2m}\frac{\eta^{-1/2}\partial_\alpha \eta^{\alpha\beta}\eta^{1/2}\partial_\beta r_{2D}}{r_{2D}}$$
$$- \frac{\hbar^2}{2m}\left((\tfrac{1}{2}\mathrm{tr}_2(\Gamma^{\alpha}{}_{3\beta}))^2 - \det{}_2(\Gamma^{\alpha}{}_{3\beta})\right) = 0. \tag{3.17}$$

We note that (3.17) is the ordinary 2D curved QHJE except for the last term. Thus the last term is regarded as an embedded effect (da Costa 1981), i.e. the embedded potential

$$V_{\mathrm{eff}}^{2D} = -\frac{\hbar^2}{2m}\left((\tfrac{1}{2}\mathrm{tr}_2(\Gamma^{\alpha}{}_{3\beta}))^2 - \det{}_2(\Gamma^{\alpha}{}_{3\beta})\right). \tag{3.18}$$

This is identified with (1.1). Furthermore we note that (3.18) vanishes for a 2D sphere on account of its symmetry.

We recall that since the surface Σ is open and we deal with a scattering problem, the intuitive correspondence between the classical and the quantum mechanics through the QHJE is guaranteed. Thus, we can go on employing this picture.

It is noticeable that the embedded potential comes from the QP. In other words, its comes through $S_R = \log R$ which corresponds to the measure part of the path integral. It is known that in the path integral method, the embedded potential comes from the measure and the ordering (Matsutani 1992a). Our result is natural and supports the agreement between the path integral method and the operator formalism.

It is worth while noting that the QP in (3.14) depends on ∂_α, $\partial_3 (= \mathrm{i}\hat{p}_3)$, q^α, and q^3 while $\partial_\alpha S$ can be regarded as the momentum along Σ and $\partial_3 S \equiv 0$. In the QHJE, $\partial_3 S$ is apparently the momentum along the normal direction p_3. However it vanishes and has no effect on the system. On the other hand, the form of the QP, which contains $\partial_3 (= \mathrm{i}\hat{p}_3)$ and q^3, fixes the embedded potential (3.18). In other words, q^3 and $\partial_3 (= \mathrm{i}\hat{p}_3)$ survives in the QP and they determine the functional form of the embedded potential (3.18). Then q^3 is regarded as a parameter of the system. After we fix its form, we make q^3 vanish.

The existence of the asymmetry in $\partial_3 S$ and p_3 agrees with the behaviour in the Dirac quantization in the submanifold quantum system. As we mentioned in the introduction, using the Dirac quantization, Ikegami *et al* (1992) studied the two constraint cases $q^3 = 0$ (D-case) and $\dot{q}^3 = 0$ ($\dot{D}$-case). The D-case contains the condition $\dot{q}^3$ to ensure consistency. According to Dirac's original work (1964), for a constraint system we will introduce the 'Dirac' bracket $[\,,\,]_{\mathrm{DB}}$ instead of the Poisson bracket $[\,,\,]_{\mathrm{PB}}$ at the classical level. When we quantize the system, we replace the classical Dirac bracket with the commutator; $[\,,\,]_{\mathrm{DB}} \to [\,,\,]/\mathrm{i}\hbar$. In the D-case, after some calculations, one obtains the equations $q^3 = 0$, $\dot{q}^3 = 0$ and $\partial_3 S_c = p_3 = 0$ at the classical Hamiltonian level. Hence these variables are excluded from the system even at the classical level. After quantizing it, one obtains an embedded potential (1.2) but it disagrees with ours (3.18). On the other hand, the $\dot{D}$-case, since its Hamiltonian does not include the normal dynamics, has $\partial_3 S_c = 0$. However the normal variables p_3 and q^3 have a physical meaning at the classical level. In other words, we can set their non-trivial Dirac brackets; $[q^3, p_3]_{\mathrm{DB}} = 1$. Hence the correspondence between $\partial_3 S_c$ and p_3 is broken there. After quantization, the algebra generated by $[\,,\,]/\mathrm{i}\hbar$ contains $[\hat{q}^3, \hat{p}_3] = \mathrm{i}\hbar$ and then one also obtains its embedded potential (1.3). In the deformation from the classical to the quantum mechanics, there is the 'ordering' problem and some ambiguity. In their paper, they chose the 'physical' ordering under which the normal kinetic operator consists of the bilinear form of the Hermite normal momentum operator $\hat{p}_3$; $\xi_2 = 2$ and $\xi_3 = 1$ in (1.3). Then the embedded potential (1.3) is in agreement with ours (3.18).

We note that in classical submanifold physics, the CHJE of our system and the Dirac bracket have an equivalent physical meaning. To see this is easy. We employ the same confinement potential $V^{2D}_{\rm conf}$. In the classical theory, the dynamics of a particle constrained on the surface Σ is obtained from the CHJE which is (3.9) without the last QP term. After a confinement limit, the dynamics along the normal direction is frozen; $q^3 = 0$, $\dot{q}^3 = 0$ and $S_{\rm cN} = 0$. Then we obtain the 2D CHJE that is (3.17) without the 2D QP term or the embedded potential (3.18). The 2D CHJE indicates the classical submanifold physics and corresponds to the Hamiltonian which is obtained by means of the classical Dirac constraint scheme (Ikegami *et al* 1992). Furthermore both the QHJE and the Dirac quantization give the method of deformation from the classical mechanics to the quantum mechanics respectively. In our argument, we formally attach the QP to the CHJE with the continuity equation. In the Dirac scheme, we replace $[\,,\,]_{\rm DB} \to [\,,\,]/i\hbar$. Then in the QHJE, $\partial_3 S = 0$ but the added QP contains q^3 and $\partial_3 (= i\hat{p}_3)$ while in the $\dot{D}$-case $\partial_3 S_{\rm c} = 0$ but q^3 and p_3 remain as dynamic variables. Thus we conclude that the asymmetry in $\partial_3 S$ and p_3 appearing in both methods can be interpreted as the same physical phenomenon. In the QHJE, the reason why $\partial_3 S$ vanishes is that the normal direction is the bound state. Accordingly, the constraint system should be regarded as a limit of the bound system. Hence the asymmetry in the Dirac scheme is very natural. Consequently in the Dirac scheme, we must choose the $\dot{D}$-case rather than the D-case. In other words, in the D-case, the condition $q^3 = 0$ is too strict to express the quantum fluctuation or the QP.

It is also worth while noting that our argument does not depend upon the exact form of the confinement potential $V^{2D}_{\rm conf}$ as long as the potential is independent of position on the surface Σ. As we mentioned in the section 2, the bound state breaks the natural correspondence between the CHJE and the QHJE. In the confinement limit, $\partial_3 S_{\rm N}$ along the normal direction becomes meaningless and only the penetration $r_{\rm N}$ is dominant. This phenomenon does not depend on the exact form of the confinement potential. For example, if we employ a box potential as a confinement potential and make its width d vanish, we obtain (3.18). For the other confinement potentials, it is also clear that $\partial_3 S_{\rm N} = 0$ and q^3 is regarded as a parameter to determine the embedded potential (3.18). It turns out that we then obtain the same 2D dynamics (3.16)–(3.18).

4. Conclusion

In the polar representation, the Schrödinger equation is related to classical mechanics or, in some situations, the CHJE. Hence, in our argument, the origin of the embedded potential in the submanifold quantum mechanics is more evident than in the ordinary methods.

By utilizing this property, we have studied the anomalous result on the Dirac constraint quantization in a submanifold; the D-case and the $\dot{D}$-case. The Dirac scheme indicates how the classical mechanics is deformed to the quantum mechanics. On the other hand, in the QHJE, the QP term indicates the deformation from the CHJE to the quantum mechanics. Both methods must be equivalent. Accordingly in order to clarify why the $\dot{D}$-case is more natural than the D-case, these cases were compared with the QHJE. In the submanifold physics, we found that though the CHJE does not contain the normal dynamics, the QP includes it and indicates the quantum fluctuation along the normal direction; it gives the embedded potential (3.18). In the $\dot{D}$-case, the classical Hamiltonian does not contain the normal dynamics, but the normal variables are maintained through the Dirac bracket. Then quantizing the system, one has the true embedded potential (1.3). Hence the structure in the $\dot{D}$-case agrees with that of the QHJE method. On the other hand, in the D-case, the normal dynamics are excluded totally and it generates an unphysical embedded potential (1.2). As Ikegami *et al* remarked, the anomalous result in the Dirac scheme is inevitable. The situation in the QHJE method shows why the $\dot{D}$-case reflects the physics. Roughly speaking, in the $\dot{D}$-case, the condition $\dot{q}^3 \propto \partial_3 S_c = 0$ restricts the Hamiltonian or the CHJE, but does not influence the QP explicitly. It expresses the physical fluctuation. However in the D-case, the conditions $q^3 = 0$ and $\dot{q}^3 \propto \partial_3 S_c = 0$, are too strict to express the QP exactly.

We have shown that the intuitive correspondence between the QHJE and the CHJE disappears for the bound state, and the submanifold quantum system should be regarded as a kind of bound system. The correspondence between the classical and the quantum case is ill defined there. Thus we conclude that the difference between the D-case and the $\dot{D}$-case comes from the ill definition. Since the QP plays a more important role there than $\partial_i S$, the condition which cannot represent the QP is fatal. In other words, on the deformation from the local to the global theory, the QP adjusts the difference and generates the quantum fluctuation. However the condition $q^3 = 0$ is global in the local (classical) theory. Hence it prevents its adjustment in the deformation. Thus the D-case is not physical at all. Consequently in the Dirac scheme, we must choose the local condition $\dot{q}^3 = 0$.

Furthermore through the polar representation, we have commented on the origins of the embedded potential (3.18) in the path integral, the operator method and the Dirac scheme respectively. The polar representation is, thus, qualified for an overview of the relations between the path integral method, the operator formalism and the Dirac scheme.

We will comment on the boundary problem. Though I did not deal with the compact submanifold, we can use (3.18) there (da Costa 1981). We must also consider the discretized condition there.

Next we mention a relevant optical problem. In optics it is known that a similar effect on a submanifold is found for a bent optical waveguide (Miyagi 1989). It is natural because there is an analogy between the quantum mechanics and wave-optics (Guillemin and Sternberg 1984). Our consideration can be applied to the optical problem.

Finally we comment on an open problem. In elementary particle physics, it is known that there are many studies of the path integral method for the Dirac constraint scheme (Senjanovic 1976, Batalin and Fradkin 1987). There are some applications of it to the submanifold system but they are just the D-case (Marinov and Terentyev 1979, Fukutaka and Kashiwa 1987). It is expected that there also appears an anomalous correspondence between the D-case and the $\dot{D}$-case. However, so far as I know, nobody has confirmed it.

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